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A Model Parameter Identification Method for Battery Applications

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ABSTRACT

Due to growing interest in hybrid and electric vehicles, the battery, being one of the critical components, is receiving a lot of attention from designers and researchers. Two battery-modeling approaches, though seemingly different, share the same mathematical challenge of robust non-linear curve-fitting. The two methods are battery equivalent circuit model and battery system level thermal modeling using the linear time-invariant (LTI) method. Both modeling approaches involve curve-fitting testing data or data from advanced models to identify four parameters in a circuit model consisting of two pairs of *RC* elements. Such curve-fitting is mathematically a non-linear least-squares (LS) problem. Standard methods like the Levenberg-Marquardt (LM) method can be used for non-linear curve-fitting, but the LM method is known to be sensitive to initial conditions. Due to the unique features of the two pairs of *RC* values in the model, the curve-fitting problem can be reformulated into a linear LS problem. Solution from the linear LS problem can then be used as an initial condition for the LM method for greater accuracy. Since the initial conditions from the linear LS problem are already close to the minimum, the sensitivity issue associated with the LM method is mitigated.

I. INTRODUCTION

Due to environmental concerns and the depletion of fossil fuels, the automotive industry has invested heavily in electric vehicles (EV) and hybrid electric vehicles (HEV). For both types of vehicles, the battery plays a critical role in affecting the overall vehicle performance. Numerical simulation becomes an indispensable tool for battery designers and researchers. Since the battery is a multi-physics application, its simulation or modeling involves many different disciplines, including but not limited to electrochemical modeling, thermal modeling, electrical circuit modeling, structural modeling, etc. Two of the modeling efforts, though seemly different, share the same mathematical problem of non-linear curve-fitting. One is the battery equivalent circuit modeling and the other is battery system-level thermal modeling.

The battery equivalent circuit model has gained popularity among system-level design engineers due to its ease of use and its capability of representing state-of-charge, I-V characteristics, and dynamic behavior of a battery system [1,2]. A commonly used equivalent circuit model consists of an open-circuit voltage source, a resistor in series, and two pairs of parallel resistor-capacitor (*RC*) elements as shown in Fig. 1. The series resistance can be determined quite easily. The main challenge is to identify the parameters of the two parallel *RC*'s through curve-fitting test data. The simplicity of the battery equivalent circuit model and yet its satisfactory results rely on good curve-fitting to test data to identify values of the two *RC*'s.

System-level thermal modeling using the linear time-invariant (LTI) approach, originated from electronics cooling applications [3,4], has recently been adopted successfully to battery cooling [5, 6, 7]. This method treats the thermal problem as an LTI system. In building a simple LTI system, two pairs of *RC* network similar to those from the battery equivalent circuit model are used [5, 6, 7]. An example of such an LTI model with two pairs of *RC* network is shown in Fig. 2. One needs to identify the values of the *RC*'s through curve-fitting the impulse or step response of the *RC* network to that of the original thermal system.

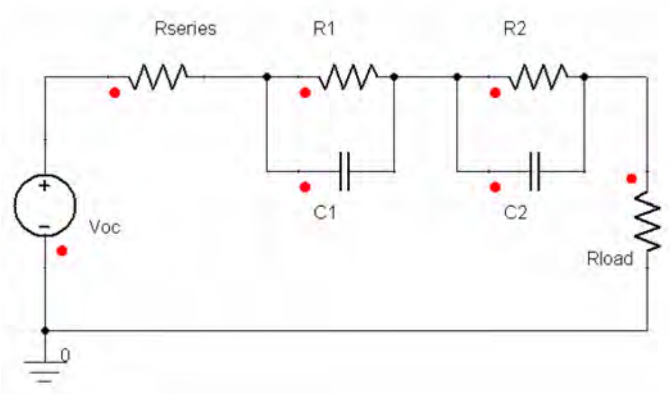


Fig. 1. A battery equivalent circuit model.

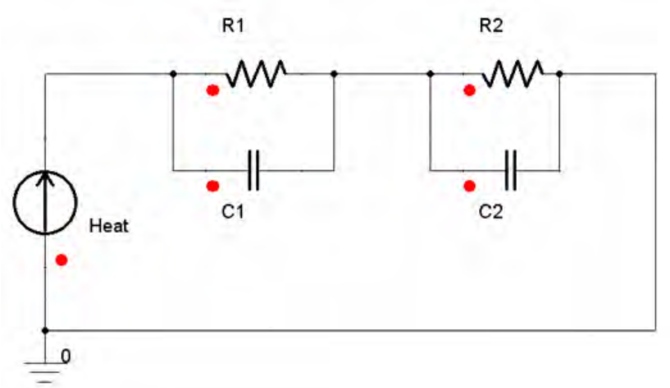


Fig. 2. An LTI model for battery thermal system.

For the two battery applications mentioned above, even though the problems addressed and physics involved are quite different, the essential mathematics involved in both cases is non-linear curve-fitting. For general non-linear curve-fitting problems, standard methods like the Levenberg-Marquardt (LM) [8,9] can be used. However, the LM method is known to be sensitive to initial conditions. Based on the unique features of the two pairs of *RC* employed in the model, Jiang [10] introduced a novel method of performing the curve-fitting by turning the non-linear curve-fitting problem into a linear one followed by solving a quadratic equation. While Jiang's method works well for a lot of cases, a few issues are observed in some applications. The current paper proposes an improved approach to resolve the observed issues.

The paper is organized as follows. Section II formally defines the mathematical problem. Section III introduces Jiang's method. Section IV demonstrates the issues associated with Jiang's method and the modifications used to improve the method. A complete solution using modified Jiang's method as an initial condition for the LM method is then proposed. This hybrid approach shows improvement over Jiang's approach and LM method used individually. Finally, Section V is the conclusion.

II. MATHEMATICAL STATEMENT OF THE PROBLEM

For models using Fig. 1 and Fig. 2, the *RC* values are determined by curve-fitting to test data or results from other simulation models. In Ref. [10], Jiang shows that the analytical solution of the *RC* circuit starting right after the pulse discharge is the following:

$$U(t) = V_{10}e^{-\frac{t}{\tau_1}} + V_{20}e^{-\frac{t}{\tau_2}} \tag{1}$$

Therefore, one needs to determine the values for parameters V_{10}, τ_1, V_{20}, τ_2 from a given set of test or simulation results. Consider a case having m sets of data $(t_1, U_1),\ldots,(t_m, U_m)$ with $m>4$. This process then becomes a non-linear least-squares (LS) problem. Formally, we are seeking a local minimizer for

$$F(\bar{x}) = \frac{1}{2}\sum_{i=1}^{m}[f_i(\bar{x})]^2 = \frac{1}{2}\left\|\bar{f}(\bar{x})\right\|^2 \tag{2}$$

where $\vec{x} = [V_{10},\ \tau_1,\ V_{20},\ \tau_2]$ and $f_i\colon \mathbb{R}^4 \rightarrow \mathbb{R}, i = 1, \ldots, m\ (m > 4)$. f_i are functions defined as follows:

$$f_i(\vec{x}) = U(\vec{x}, t_i) - U_i \tag{3}$$

where $U(\vec{x}, t_i)$ is defined in Equ (1) with $\vec{x}$ being $[V_{10}, \tau_1, V_{20}, \tau_2]$. For a general non-linear curve-fitting problem, namely a general function instead of the specific form of Equ (3), the standard LM method [8-9] can be used. However, the performance of the LM method is sensitive to initial conditions, which are necessary since the LM method involves iteration. (The LM method is described briefly in Appendix A for reference.)

Another general method is to perform the curve-fitting in the frequency domain using vector-fitting (VF) [11,12]. This approach involves identifying the transfer function of the proposed model with the sampled Fourier transform of the impulse response of the modeled system. While this is a very powerful and general technique, it is more involved to implement.

Because of the specific form of function *U* shown in Equ (1) and salient features of test results, certain techniques have been developed for this non-linear curve-fitting problem for battery equivalent-circuit models. A commonly used method makes use of the property that each of the two time constants plays a dominant role at different stages of the battery voltage response, and calculates the model parameters accordingly [13]. For such a method, the process is quite manual and the results are greatly influenced by the partition of faster and slower dynamics of the battery and the selection of the data points used for the calculation. This method does not apply to battery thermal LTI modeling since the required data property is not possessed by battery thermal systems.

In this paper, a simple and novel method of parameter identification is presented. It makes use of a regression equation which is linear in variables that can be measured or calculated from the test or other advanced models. Such a method was originally proposed by Jiang [10]. While Jiang's method works well for a lot of cases, there are issues observed in some cases. Modifications are made to Jiang's method in this paper to overcome the observed issues. After the solution from the modified Jiang's method is obtained, it is used as the initial condition for the LM method. The LM method is used because results from the modified Jiang's method do not satisfy the LS solution of the original problem, namely Equ (2). With the combination of the modified Jiang's method and the LM method, the final solution satisfies the LS problem and the approach becomes less sensitive to the initial condition since the initial condition is already close to the optimum solution.

III. JIANG'S METHOD FOR PARAMETER IDENTIFICATION

The starting point of Jiang's method is two functions, X and Y, defined by the following two equations:

$$X = \int_0^t U(\tau) \cdot d\tau \tag{4}$$

$$Y = \int_0^t X(\tau) \cdot d\tau \tag{5}$$

Substitution of U from Equ (1) into Equ (4) and (5) gives the following expressions for X and Y, respectively,

$$X = (V_{10}\tau_1 + V_{20}\tau_2) - \left(V_{10}\tau_1 e^{-t/\tau_1} + V_{20}\tau_2 e^{-t/\tau_2}\right) \tag{6}$$

$$Y = (V_{10}\tau_1 + V_{20}\tau_2)t + \left(V_{10}{\tau_1}^2 e^{-t/\tau_1} + V_{20}{\tau_2}^2 e^{-t/\tau_2}\right) - (V_{10}{\tau_1}^2 + V_{20}{\tau_2}^2) \tag{7}$$

Solving for $e^{-t/\tau 1}$ and $e^{-t/\tau 2}$ using Equ (1) and (6) and substituting the results into Equ (7) gives the final expression for Y,

$$Y = (\tau_1 + \tau_2)(-X) + (\tau_1\tau_2)(-U) + (V_{10}\tau_1 + V_{20}\tau_2)t + (V_{10} + V_{20})\tau_1\tau_2 \tag{8}$$

When the m sets of experimental data points of (t_i, U_i) are used to evaluate t, U, X, and Y in Equ (8), one obtains m equations with four unknowns. These m equations can be written in a matrix form shown below,

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_m \end{bmatrix} = \begin{bmatrix} -X_1 & -U_1 & t_1 & 1 \\ -X_2 & -U_2 & t_2 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ -X_m & -U_m & t_m & 1 \end{bmatrix} \cdot \begin{bmatrix} \tau_1 + \tau_2 \\ \tau_1\tau_2 \\ V_{10}\tau_1 + V_{20}\tau_2 \\ (V_{10} + V_{20})\tau_1\tau_2 \end{bmatrix} \tag{9}$$

The solution to the above equation is obtained by solving a linear LS problem. After Equ (9) is solved, V_{10}, τ_1, V_{20}, τ_2 can be obtained by solving a quadratic equation for τ_1 and τ_2 and a linear set of two equations for V_{10} and V_{20}. It is interesting to note that a non-linear LS problem has been changed to a linear LS problem using Jiang's approach.

IV. MODIFICATION OF JIANG'S METHOD

There are a few problems observed using Jiang's method in some cases. First of all, though not often, the matrix in Equ (9) can be rank deficient for certain cases. This issue can be solved quite easily by using singular value decomposition (SVD) based LS solver rather than QR-based LS solver. So, all linear LS problems solved in this paper use the SVD-based LS solver. When the matrix is not rank deficient, the SVD-based LS solver gives the same results as the QR-based solver. When the matrix is rank deficient, the SVD-based solver is stable and gives a least-squares and least-norm solution.

Secondly, the method only ensures a good fit for the function Y, not the original function U that is ultimately needed. In some cases, a good fit for the function Y does not translate to a good fit for the function U. Large errors of more than 100% (in the sense of 2-norm) have been observed for the function U even when the error for Y is quite small. Fig. 3 and 4 show such an example. It is clear from Fig. 3 that the fit for Y is quite good. However, the fit for U shown in Fig. 4 is rather poor near t = 0 as indicated in Fig. 4b). This result can be explained if one notices that the area difference between the two curves in Fig. 4 is quite small even though the values of the two curves near t = 0 are quite different. But Jiang's method essentially compares the area rather than the values for the function U. This is clear from how X and Y are defined in Equ (4) and (5).

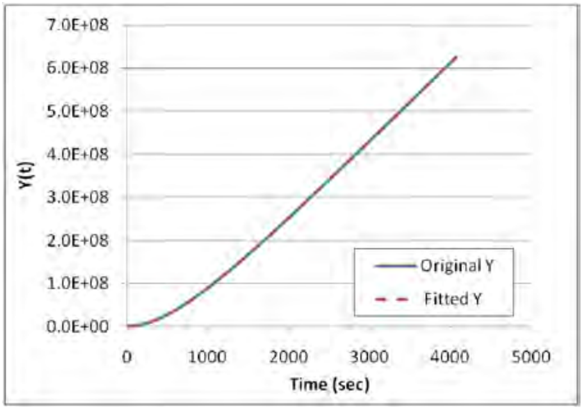
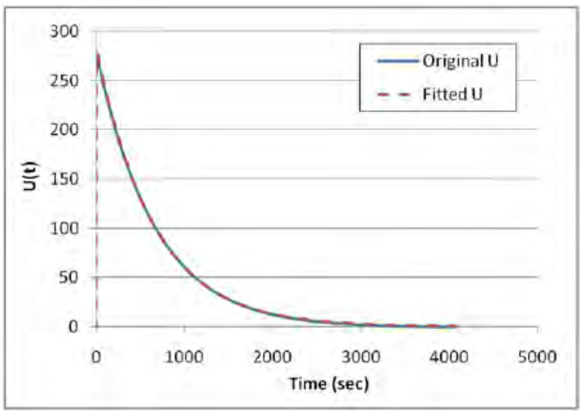
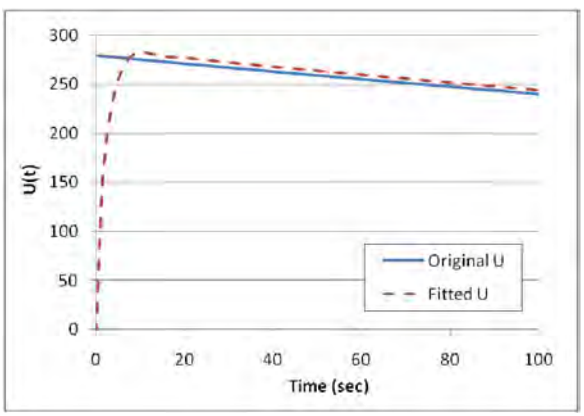


Fig. 3. A result for function Y using Jiang's method



a). Zoomed-out view



b). Zoomed-in view near t = 0

Fig. 4. A result for function U using Jiang's method.

This issue can be resolved quite nicely with one observation. From Equ (1), it is clear that $V_{10}+V_{20} = U_0$, where U_0 represents the function U at $t = 0$. After this relation is substituted into Equ (8), one obtains the following new expression for Y,

$$Y = (\tau_1 + \tau_2)(-X) + \tau_1\tau_2(U_0 - U)$$
$$+(V_{10}\tau_1 + V_{20}\tau_2)t \quad (10)$$

By using Equ (10) rather than Equ (9), we are forcing the sum of V_{10} and V_{20} to be the correct value of U_0. This helps avoid the issue observed near $t = 0$ associated with Jiang's original approach. The corresponding matrix form for Equ (10) is the following:

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_m \end{bmatrix} = \begin{bmatrix} -X_1 & U_0 - U_1 & t_1 \\ -X_2 & U_0 - U_2 & t_2 \\ \vdots & \vdots & \vdots \\ -X_m & U_0 - U_m & t_m \end{bmatrix} \cdot \begin{bmatrix} \tau_1 + \tau_2 \\ \tau_1\tau_2 \\ V_{10}\tau_1 + V_{20}\tau_2 \end{bmatrix} \quad (11)$$

With the above modification, the fitting results for the functions Y and U shown in Fig. 3 and 4 are repeated in Fig. 5 and 6, respectively. Visually, there is no difference for the fitting of the function Y using the original method and the modified one, but the improvement for the more important function U is quite obvious near $t = 0$ as can be seen by comparing Fig. 6 with Fig. 4.

Another observation with Jiang's method is accuracy. Results from the method do not satisfy Equ (2), so it is not a LS solution to the problem. In order to obtain the more accurate LS solution, the LM method is used with the initial conditions from the above modified Jiang's method.

It is worthwhile to mention that Jiang's approach only works well with two RC pairs. When more than two pairs are used, the counterparts of Equ (10) and (11) become too complicated to be useful.

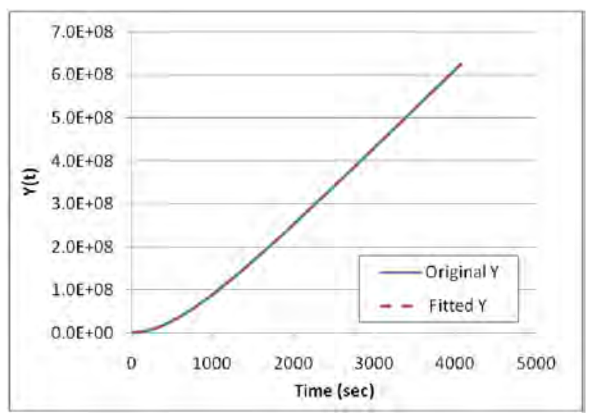
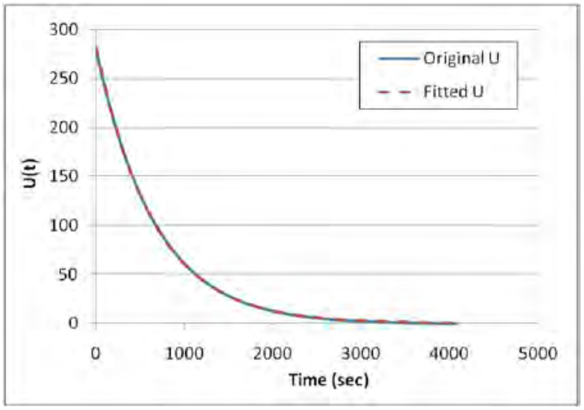
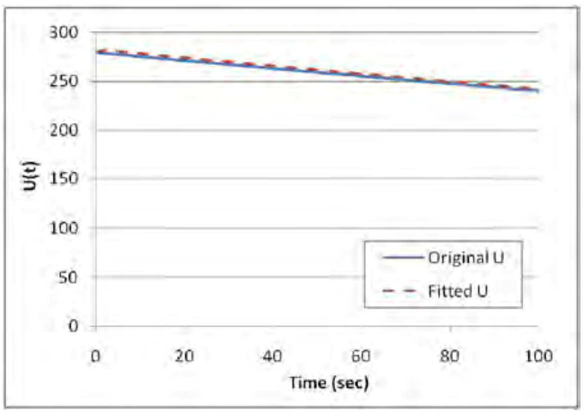


Fig. 5. A result for function Y using modified Jiang's method.



a). Zoomed-out view



b). Zoomed-in view near t = 0

Fig. 6. A result for function U using modified Jiang's method.

As a final observation, both the original and modified Jiang's method could generate complex (as opposed to real) solutions for V_{10}, τ_1, V_{20}, τ_2 after the linear LS problem is solved from Equ (9) or (11). It was observed that this happens more often when the step response data does not reach steady-state. But it can occur even when the step response data approach steady-state, and this is especially true when the curve to be fitted looks like the one shown in Fig. 7. While such a curve may not show up in the battery circuit model, it shows up in the LTI thermal model when cross-heating is involved. The author has not found a method to resolve this issue within the framework of Jiang's method. For such a curve, one can use the LM method directly and cope with the sensitivity issue. Results from LM method using two rather randomly selected initial conditions are shown in Fig. 8 to demonstrate the sensitivity issue. The first initial condition gives an error of 2.5% while the second gives an error of 7.6% (based on the 2-norm in both cases). Note that even the worse result of using initial condition 2 might be acceptable since cross-heating makes a small contribution to the final solution and the curve-fitting accuracy is thus less important for these curves from cross-heating. If higher accuracy is desired, though, the more powerful vector-fitting method [11] in the frequency domain can be used. For the curve shown in the Fig. 7, fitting results from the vector-fitting method generate a good fit as shown in Fig. 9. Such a method is robust and does not suffer the sensitivity issue associated with time-domain fitting using the LM method. But the vector-fitting method is more involved. Frequency-domain fitting is beyond the scope of this paper and will not be discussed further here.

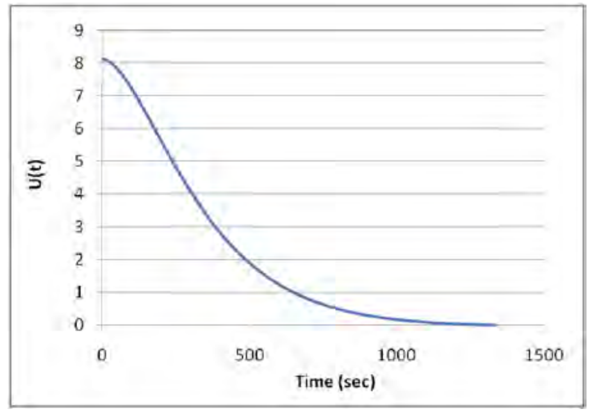


Fig. 7. A function U due to cross heating that results in complex V_{10}, τ_1, V_{20}, τ_2 using Jiang's method or the improved version.

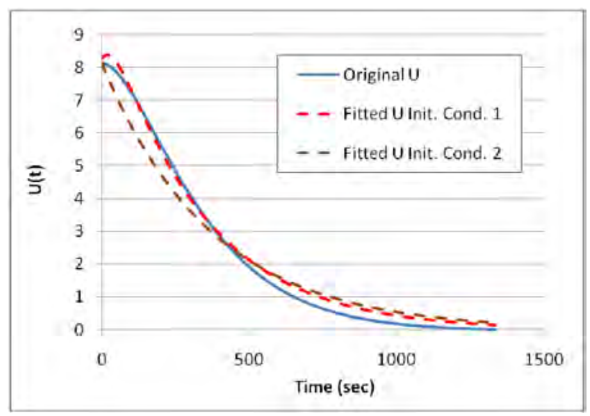


Fig. 8. Fitting results using LM method for two different initial conditions.

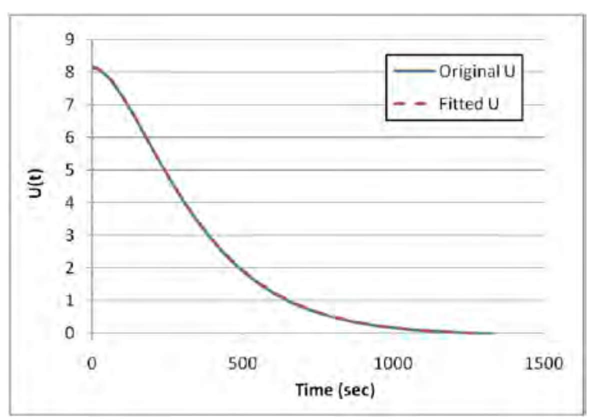


Fig. 9. Fitting results using vector-fitting method.

The results shown so far are all for battery thermal LTI modeling. As mentioned before, this approach works just as well for battery equivalent circuit modeling. Fig. 10 shows curve-fitting results for the function *U* calculated from a battery pulse discharge using the Newman electrochemistry model [14, 15, 16]. Fig. 11 shows curve-fitting results for the function *U* provided by A&D Technology from test results. Note that the values for *U* to be fitted can be generated either from more advanced models or from test data.

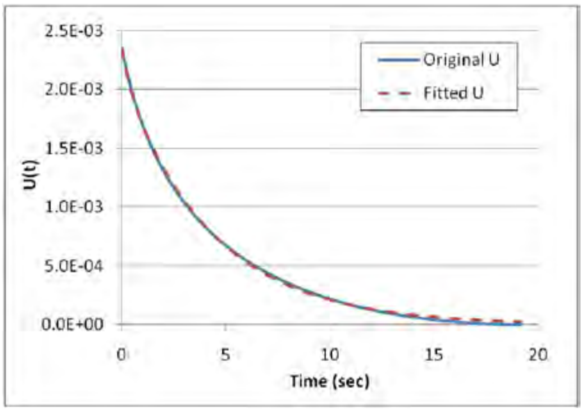


Fig. 10. A result for function U using the modified Jiang's method. Original U values are obtained from the Newman electrochemistry model.

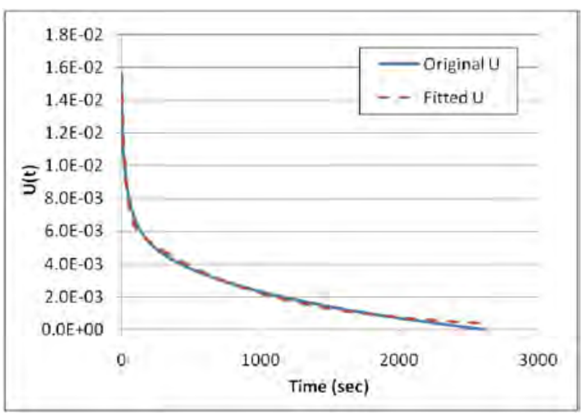


Fig. 11. A result for function U using the modified Jiang's method. Original U values are provided by A&D Technology.

V. CONCLUSION

Two battery applications share the same mathematical challenges of non-linear curve-fitting for parameter identification. Non-linear curve-fitting can be solved using the standard LM method, but the LM method is sensitive to initial conditions. A novel method which makes use of a regression equation that is linear in the variables that can be measured or calculated from testing or from other advanced models is proposed for the parameter identification. Results from this method are then used as the initial condition for the more accurate LM method. The combined approach makes the method less sensitive to initial conditions compared with using the LM method alone and more accurate than without using the LM method.

The approach however is not without issues. The main problem observed is that the parameters could be complex for battery cell cross-heating curves. Fortunately, accurate fitting for these curves is not as critical since cross-heating usually does not contribute significantly to the final temperature solution. If high accuracy is desired, a good fit can be achieved by using frequency domain fitting methods, though such methods are more involved to implement.

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APPENDIX

Appendix A: Levenberg-Marquardt Method for Battery Curve-Fitting

Given a vector function $\vec{f}: \mathbb{R}^n \rightarrow \mathbb{R}^m$ *with m > n*. We are seeking a local minimizer for

$$F(\vec{x}) = \frac{1}{2}\sum_{i=1}^{m}[f_i(\vec{x})]^2 = \frac{1}{2}\left\|\vec{f}(\vec{x})\right\|^2 \tag{A1}$$

For the battery application, the vector function is $\vec{f}: \mathbb{R}^4 \rightarrow \mathbb{R}^m$, ($m > 4$), whose components are defined as follows:

$$f_i(\vec{x}) = c_1 e^{-\frac{t_i}{\tau_1}} + c_2 e^{-\frac{t_i}{\tau_2}} - y_i \tag{A2}$$

where $\vec{x} = [c_1, \tau_1, c_2, \tau_2]$, and ($t_i$, y_i) are testing data or data from other models at m different points,

A linear approximation for $\vec{f}$ in the neighborhood of $\vec{x}$ for a small $\vec{h}$ is the following:

$$\vec{f}(\vec{x}+\vec{h}) \approx \vec{f}(\vec{x}) + \vec{\vec{J}}(\vec{x}) \cdot \vec{h} \tag{A3}$$

where $\vec{\vec{J}}$ is the *Jacobian* defined by:

$$\vec{\vec{J}}(\vec{x}) = \nabla\vec{f}(\vec{x}) \tag{A4}$$

Inserting Equ (A3) into Equ (A1) gives the following:

$$F(\vec{x}+\vec{h}) \approx F(\vec{x}) + \vec{h}^T\vec{\vec{J}}^T\vec{f} + \frac{1}{2}\vec{h}^T\vec{\vec{J}}^T\vec{\vec{J}}\vec{h} \tag{A5}$$

For a given $\vec{x}$, Equ (A5) is a function of $\vec{h}$. This function is defined as $L(\vec{h})$, shown below,

$$L(\vec{h}) = F(\vec{x}) + \vec{h}^T\vec{\vec{J}}^T\vec{f} + \frac{1}{2}\vec{h}^T\vec{\vec{J}}^T\vec{\vec{J}}\vec{h} \tag{A6}$$

The gradient of L is the following:

$$\nabla L(\vec{h}) = \vec{\vec{J}}^T\vec{f} + \vec{\vec{J}}^T\vec{\vec{J}}\vec{h} \tag{A7}$$

Setting $\nabla L(\vec{h})$ equals to zero provides a mininizer. This is called the Gauss-Newton method, and the resulting equation is the following:

$$\left(\vec{\vec{J}}^T\vec{\vec{J}}\right)\vec{h}_{gn} = -\vec{\vec{J}}^T\vec{f} \tag{A8}$$

The above equation is equivalent to the following LS problem. Numerically, it is better to solve the following LS problem rather than (A8).

$$\vec{\vec{J}} \cdot \vec{h}_{gn} = -\vec{f}$$

(A9)

In the LM method, it is suggested to use a damped Gauss-Newton method. The step is defined by the following modification to Equ (A8)

$$\left(\vec{\vec{J}}^T \vec{\vec{J}} + \mu I\right) \vec{h}_{lm} = -\vec{\vec{J}}^T \vec{f}$$

(A10)

When μ is very small, then $\vec{h}_{lm} \approx \vec{h}_{gn}$, and the LM method goes back to the Gauss-Newton method. When μ is large,

$$\vec{h}_{lm} \approx -\frac{1}{\mu}\vec{\vec{J}}^T \vec{f} = -\frac{1}{\mu}\nabla F$$

(A11)

i.e. a short step in the steepest descent direction. So, the LM method can be thought of as a combination of the steepest descent and the Gauss-Newton method. When the current solution is far from the minimum, the algorithm behaves like the steepest descent method, which is slow in convergence but guaranteed to converge. When the current solution is close to the minimum, it becomes the Gauss-Newton method.

The value for μ has to be modified based on the solution from the current iteration. The choice of initial μ value is not critical as the solution is not sensitive to the initial value. During the iteration the size of μ can be updated by the *gain ratio* defined below,

$$\rho = \frac{F(\vec{x}) - F\left(\vec{x} + \vec{h}_{lm}\right)}{L(0) - L\left(\vec{h}_{lm}\right)}$$

(A12)

A large value of ρ indicates that $L(\vec{h})$, is a good approximation to $F(\vec{x} + \vec{h})$, and the damping may be reduced. A small value of ρ, including negative values, indicates that we should increase the damping factor and thereby increase the penalty on large steps. It was demonstrated in Nielsen [17] that the following strategy in general works well,

if $(\rho > 0)$

$\mu = \mu \times max\{1/3, 1 - (2\rho - 1)^3\}; v = 2;$

else

$\mu = \mu \times v; v = 2 \times v;$

A detailed description of the LM method can be found in [9-10, 17]. The algorithm can be summarized in the following pseudo-code.

Input: m sets of testing data (t_i, y_i),
$\vec{f}: \mathbb{R}^n \to \mathbb{R}^m \; with \; m > n$, and $\vec{x}_0$.
$\vec{x}_0$ is from modified Jiang's method in this case.
Output: $\vec{x} \in \mathbb{R}^n$ that minimizes $\left\|\vec{f}(\vec{x})\right\|^2$

begin
$k = 0;\ v = 2; \vec{x} = \vec{x}_0;$
$fit\_error = \sqrt{2 \cdot F(\vec{x})}/\|\vec{y}\|$
found = $fit\_error < \varepsilon$; assign initial μ value
while(**not** *found*) **and** $(k < k_{max})$
$k = k + 1;$
if$(\mu \approx 0)$
Solve $\vec{\vec{J}} \cdot \vec{h}_{lm} = -\vec{f}$ for $\vec{h}_{lm}$
else
Solve $\left(\vec{\vec{J}}^T \vec{\vec{J}} + \mu I\right) \vec{h}_{lm} = -\vec{\vec{J}}^T \vec{f}$ for $\vec{h}_{lm}$
$\vec{x}_{new} = \vec{x} + \vec{h}_{lm}$
$\rho = \left(F(\vec{x}) - F(\vec{x}_{new})\right) / \left(L(0) - L(\vec{h}_{lm})\right)$
if$(\rho > 0)$
$\vec{x} = \vec{x}_{new}$
$fit\_error = \sqrt{2 \cdot F(\vec{x})}/\|\vec{y}\|$
found = $fit\_error < \varepsilon$;
$\mu = \mu \times max\{1/3, 1 - (2\rho - 1)^3\};\ v = 2;$
else
$\mu = \mu \times v;\ v = 2 \times v;$
return $\vec{x}$
end

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